# GEOMETRICAL TRANSFORMATIONS – CONSTRUCTIVIST ANALYTIC APPROACH

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## ABSTRACT

The contribution illustrates a constructivist approach to the teaching of geometrical transformations to future mathematics teachers at the Faculty of Education, Charles University in Prague. Traditionally, this subject was presented as a series of logically connected definitions and theorems and students were asked to apply them in problems. A lot of material was covered like this, however, students' understanding was often formal and superficial. Several years ago, the course was completely re-designed in such a way as to let students deduce most knowledge themselves through a series of carefully prepared problems. A textbook adopting the Klein approach to geometry was written for the course (in Czech). Only isometries and affine transformations in the line and plane were covered, however, our experiences show that the investigative approach leads to a better understanding of the subject matter and improves students' ability to study transformations independently of the teacher.

A year ago, the author taught geometrical transformations in English to a group of practising teachers and the course was refined. Where it was possible, no mathematical result was presented as a ready made product, students had to discover it for themselves. As the analytic approach to transformations lends itself to using software (e.g. *Maple*), the emphasis was placed on its use to help with tedious calculations. The article concentrates on the basic characteristics of the course: emphasis on the connection between synthetic and analytic approaches, connections between geometry and algebra, investigative learning, use of computer and non-traditional assessment. An illustration is given of a student's investigation of the general matrix for a glide reflection. Examples of problems for the final test are discussed.

**Keywords:** constructivist approach, investigation, analytic and synthetic geometry, *Maple*, geometrical transformations, isometry, affine transformation

# 1. Introduction

In the traditional (and prevailing) teaching of university mathematics, we often try to pass as much knowledge as possible to students and present “the finished and polished product into which that well known, unassailable, fully accepted segment of mathematics has grown” (Dreyfus, 1991). However, this does not necessarily mean that students understand the mathematics they are being taught. Their knowledge is often formal.

In the nineties, research in mathematics education (not only) in the Czech Republic has taken into account constructivist approaches, which are gradually finding their way to the teaching of mathematics at the primary and secondary school (e.g. Hejny, Kurina, 2001, Jaworski, 1994). However, as far as we know the instances of using the constructivist way of teaching at the university level have been rare. Moreover, we realised that when student teachers are prevented from experiencing constructivist approaches during their university study, they can hardly be expected to use them in their own teaching. Therefore, we attempted to remedy the situation and redesigned the course of analytic geometry. Here we will concentrate on the part of the course which focuses on geometrical transformations.

# 2. The course of analytic geometry - history

A course on analytic geometry has always had its place in the preparation of future mathematics teachers at the Faculty of Education, Charles University in Prague[1]. It used to be given in a traditional form: 'definitions - theorems - proofs - exercises'. In 1995, Prof. Hejny redesigned the course so that it better reflected constructivist teaching. It meant, among other matters, markedly cutting down on the content of the course and presenting the content at a less advanced level and in greater detail than was customary. A university textbook (Hejny, Jirotkova & Stehlikova, 1997) was prepared in which more stress was put on student investigations. Most theorems emerge only as a result of a series of carefully selected problems; some of them must be formulated and proved by students themselves. It must be stressed that the textbook is unsuitable for the use as a reference book (it is far too ‘chaotic’), it cannot be read, it only can be studied. It also requires a teacher who is prepared to teach in a constructivist way.

The author of this paper has used the textbook for four years at the Faculty of Education and later in a course for practising teachers at a foreign university. This enabled her to further reflect on the course and the way it is delivered, and to modify it. Here we will concentrate on this modification.

# 3. The goal of the course and its outline

The course main goal is **not** to teach students as many different concepts, definitions and theorems as possible and to show them a finished ‘building’ of Euclidean and affine geometry, but rather to open the world of geometrical transformations to them and to make them aware of methods they can use for their own study of transformations. It is hoped that the course will make the subject more engaging and meaningful for them.

The course assumes a basic knowledge of isometries and similarities (taught earlier in the course of synthetic geometry) and of group theory and linear algebra (matrices). It starts with the

[1] In the Czech schools, geometry is given relatively more attention than abroad.

geometry of the Euclidean line and plane, which is well known to students, and progresses to affine geometry by extending the group of isometries into the affine group (in plane).

***Course outline***

1. Isometries in $E^1$ (Euclidean line): translation, symmetry. Synthetic and analytic views (equations). Products of isometries in $E^1$.
2. Revision of isometries $E^2$ (Euclidean plane) from the point of view of synthetic geometry: basic properties, algebra of isometries, and decomposition into the product of reflections.
3. Isometries in $E^2$ preserving the origin of the co-ordinate system. Their analytic description via matrices. Parallel between the multiplication of matrices and product of isometries.
4. Group of isometries, synthetic and analytic view. Its subgroups. Group generators.
5. All isometries in $E^2$, their matrices. Product of isometries. Inverse isometries.
6. The group of affine transformations in $A^1$ (affine line). Matrices of affinities. Products of affinities.
7. Affinities in $A^2$ (affine plane). Geometric interpretation of a matrix of an affine transformation.
8. Classification of affinities in $A^2$. Invariant points and invariant lines. Lines of self-corresponding points.
9. Affinities with a line of self-corresponding points. Perspective affinities. Shear, oblique reflection. Euclidean and affine plane. Metric properties and affine properties.
10. Decomposition of affinities into the product of affinities with a line of self-corresponding points.
11. Similarity – a synthetic and analytic view.

# 4. Main characteristics of the course

## 4.1 Emphasis on the synthetic and analytic approaches to transformations

Transformations are treated both from the synthetic and analytic way and when possible, problems are solved in these two ways. Students are encouraged to compare the suitability of the first or second approach for certain types of problems.

When investigating isometries, students start from their geometric characterisation and proceed to their analytic (matrix) description. With affine transformations, the process is reversed. Students start with a matrix of affine transformation (see below) and look for the geometric characterisation of the transformation which it represents. By a geometric (synthetic) characterisation, we mean determining some properties of the transformation, such as the properties it preserves, what its fixed points and fixed lines are, etc.

## 4.2 Emphasis on the connection between geometry and group and matrix algebra

We adopt the Klein approach to geometry, i.e. that geometry can be thought of in terms of a space and of a group acting on it. Moreover, in agreement with Schattschneider (1997), we consider it important to use the study of transformations for the visualisation of the abstract concept of a group and also for "de-emphasising number systems as examples of groups, allowing students to see that not every group has all the nice properties of number systems".[2]

[2] Schattschneider (1997) suggests using the program *Geometer's Sketchped* for the visualisation of isometries and similarities. In the course of synthetic geometry which precedes the course in question on analytic geometry, *Cabri geometrie* is used for the same purpose at our faculty.

When we consider geometries in this way, it is often convenient to have an algebraic representation for the transformations involved. This not only enables us to solve problems in geometry **algebraically**, but also provides us with formulas that can be used to compare different geometries.

In the course, we use matrices for isometries and affine transformations. However, at the beginning of the course when isometries in $E^1$ and isometries in $E^2$ preserving the origin are studied, only equations are used because there is actually no need for matrices. They come to the fore only when isometries not preserving the origin begin to be studied. Unlike most textbooks, which use only equations for transformations or 2 x 2 matrices, we use matrices 3 x 3:

Isometries: $\begin{pmatrix} a & b & c \\ \pm(-b) & \pm a & d \\ 0 & 0 & 1 \end{pmatrix}$, where $a^2 + b^2 = 1$. Affinities: $\begin{pmatrix} a & b & i \\ c & d & j \\ 0 & 0 & 1 \end{pmatrix}$, where $ad - bc \neq 0$.

### 4.3 Investigative learning

While investigative learning in primary and secondary schools is quite common, it is, in our opinion, undervalued at university level. If it is used at all, then this is usually in problem solving courses. Some tutors believe that most concepts of abstract mathematics are inaccessible to students in this way and even if students could discover them, it would take too much time. However, we believe that this time is not wasted and that the insight students get from their own investigative work is more valuable than acquiring the knowledge of many concepts introduced to them as ready-made products. The understanding and skills the students acquire by investigative learning makes up for the reduction in the content covered in the course.

In the course of analytic geometry, students are asked to derive knowledge for themselves. For instance, instead of being told what the general matrix for rotation (of α about the point $(p,q)$) is and then asked to try some examples, they have to deduce it themselves on the basis of their knowledge of the properties of rotation. Similarly instead of being told the basic theorems of affine geometry, they are asked to explore several concrete matrices of affinities and their properties and then to formulate theorems and prove them (such proofs are usually easier for them as they can use their experience from the previous experiments). Thanks to the use of 3 x 3 matrices, students cannot easily find the answers in the textbooks.

### 4.4 Use of a computer (*Maple*)

Nowadays, mathematical computer programs like *Mathematica* or *Maple* play an important role in the teaching of mathematics at university level. Many courses make use of them, especially calculus courses (e.g. Brown, Porta & Uhl, 1991, Devitt, 1993, many contributions in the Proceedings of ICTM, 1998). For geometry, *Geometer's Sketchpad* (e.g. Schattschneider, 1997, Parks, 1997) or *Cabri geometrie* (e.g. Dreyfus, Hillel & Sierpinska, 1999) are mostly used. Some research on the use of technology in advanced mathematics has been summarised in Dubinsky & Tall (1991).

As taught originally, some parts of the course of analytic geometry caused problems. The calculations, which were required to enable a student to deduce a matrix for a certain transformation, or to find the product of several transformations, were long and tedious. Therefore, in the modified course, the stress was put on the use of *Maple* as a means of helping a student to concentrate more on the overall strategy rather than on the calculation itself. The tutor started to use *Maple* herself for this purpose and produced *Maple* worksheets for the students which (projected by a data projector) formed the basis of the class work. The tutor's notes were sent to the students each week both to revise what had been done in class and to work on new problems.

Here we would like to illustrate our strategies using the example of the general matrix for a glide reflection. In the original course, it was virtually impossible to ask students to deduce this matrix and later to interpret the matrix geometrically. Thus, the teacher usually asked them to find one particular example and interpret it and supplied them with the geometric interpretation herself. The use of *Maple* enabled us to ask students to carry out the whole procedure themselves.

***Illustration – How to find the general[3] matrix for a glide reflection and conversely, how to interpret a matrix for a glide reflection geometrically***

Students know from synthetic geometry that glide reflection is the product of a reflection in a line and translation. We assume that earlier in the course they found the matrix for reflection in a line with an inclination α and the matrix for translation. Later, they are asked to find the matrix for a glide reflection and interpret it. The process has three parts (the following headings represent the tasks given to students, the text underneath is a student's solution). The figures can be found in the appendix.

**1**. ***Find the matrix for a glide reflection***

It can be done by multiplying (in any order) a matrix of reflection in a line with inclination α and a matrix of translation through vector **u**[*k* cosα, *k* sinα] (vector **u** must be parallel to the line of reflection) and simplifying the calculations (*Maple* result is given in fig. 1, α is the inclination of the line of reflection, *u, v* are co-ordinates of any point on the line of reflection, *k* is any real number).

**2**. ***How do we distinguish a matrix for reflection and a matrix for glide reflection?***

The matrix in fig. 1 is the same as the matrix for reflection in line (fig. 2) in that when we get a matrix of isometry of the form $\begin{pmatrix} a & b & m \\ b & -a & n \\ 0 & 0 & 1 \end{pmatrix}$, we cannot decide immediately which matrix it is. We must use the properties of both isometries to be able to make a decision. Unlike glide reflection, reflection in a line has a line of fixed points. So using the general matrix G in fig. 2, we compute the fixed points (it is a standard procedure for students by this stage of the course). We get a system of two equations and using knowledge from algebra[4], conclude that the system is solvable (i.e. there exist fixed points and the matrix must be the matrix of reflection in a line) iff $d = n\sin\alpha + m\cos\alpha = 0.$ Otherwise, i.e. if $d \neq 0$ , it is a matrix of glide reflection.

**3.** ***Given the matrix in fig. 2 (i.e. we know α, m, n), interpret it geometrically***.[5]

The task is to find out the line of reflection and the vector of translation. We will write down two equations (which we get by comparing the matrix in fig. 2, which can be both a matrix for a line reflection and glide reflection, and the matrix in fig. 1, which is a matrix for a glide reflection) and solve them in terms of *v* and *k*. In fig. 3, the process of determining the equation of the line of reflection and the co-ordinates of the vector is illustrated.

## 4.5 Non-traditional assessment

From the very beginning, we felt that a new type of course also required a new type of assessment. The traditional way of assessment used to be a written test comprising problems, definitions and possibly theorems and students could only use a calculator. Students very often learnt the content of the course by heart and were only able to solve standard types of problems.

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[3] In the following text, we will omit the word 'general'.

[4] All the calculations are done in *Maple*, however, due to the limited space we cannot illustrate everything.

[5] Prior to this general problem, students are asked to interpret one particular matrix geometrically, which makes the general considerations easier.

Prof. Hejny proposed a change of the form of the test such that now the students can use any aids they wish, including their notes from the course, textbooks, computers, etc. (but they must work independently). This form, however, puts greater demands on the tutor and the types of problems he/she has to prepare. They cannot be mere variations of problems solved during the course but on the other hand, students must be able to solve them using the knowledge and skills they acquired in the course. The first sets of problems were prepared by Prof. Hejny, later the author contributed problems too. Below there are three illustrations of problems from the test.

1. Let ABC be an isosceles triangle with the orthocentre $O$ and the basis $|AB| = 4$. Let us denote $u = AC, v = BC, w = AB$. Let $p$ be a line. We know that the following properties hold: $(s_u s_v)^3 = h_C, s_u s_p = s_p s_v, s_p(s_w(O)) = Q$. Find the distance $|OQ|$. Find all solutions.
2. Given a triangle *KLM* and points *N* (a midpoint of *L* and *M*), *O* (a midpoint of *K* and *M*), and *P* (a midpoint of *L* and *K*). An affine transformation *f* is given by *f*(*LPN*) = *OKP*. Express *f* as a composition of *f* = *tg* where *t* is a translation and *g* is an oblique reflection (it is sufficient to find one solution). Find fixed lines of *f*.
3. Describe via matrices a group *G* generated by three reflections in lines $x - y = 1, x - y = -1, x + y = 2$.

In the first problem, the students have to use knowledge from synthetic geometry of the basic properties of isometries. They must know how to compose them and how to work with transformational equations. It is necessary to draw a picture. The analytic approach is counter-productive here, the calculations are far too complicated.

The second problem combines synthetic and analytic approaches. This requires a lot of experimenting. Students must know how to find the object point for oblique reflection and translation. For the second part, they must introduce a co-ordinate system to be able to determine the matrix of *f* and find its fixed lines.

The third problem is best solved in an analytic way from the very beginning. Note that this task is not one, which asks students merely to verify that a certain structure is a group, but rather to generate a group, which includes certain objects.

We must stress that allowing the students to use any material during the test was at first[6] an inhibiting factor for them. Some of them thought that no studying was needed prior to the exam because they would be able to find the answers in their notes or in textbooks! This meant that they were very surprised by the problems, which they were asked to solve. They claimed "it is unfair because we did not do such problems in class". Only later, when they did study for the exam, solved problems given in class, etc. could they see that in order to solve the problems in the test, they just had put all the pieces of knowledge gained from the course together.

### 4.6 Connection with other approaches to geometrical transformations

The approach we have chosen for the course and the use of 3 x 3 matrices for transformations means that students cannot find answers to problems easily in other textbooks. Later in the course, they are encouraged to use other books as well and to see how other authors' approaches differ or are similar to the approach in the course. For instance, while in our course affine transformations are divided according to the number of fixed points and all considerations evolve from the idea of perspective affinities[7], in Gans (1969) the central concept is primitive transformations. The

[6] Later, they shared their experience with other students who subsequently did not underestimate the exam quite as much!

[7] It is because perspective affinities can be studied in a synthetic way relatively easily and every affine transformation can be decomposed into two perspective affinities.

comparison makes students aware that there is not a single 'ideal' approach to teaching mathematics and that different approaches have their advantages and drawbacks.

# 5. Conclusions

We have shown that it is possible even at the university level to teach some parts of curriculum in a constructivist way provided that we cut down on the content and stress a student's independent work. We are aware that it would be too time consuming and in some cases impossible to use this type of teaching in all subjects. However, we believe that it is worth doing at least in some courses and especially so in the preparation of future mathematics (and elementary) teachers.

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**REFERENCES**

- Brannan, D.A., Esplen, M.F., Gray, J.J. (2000). *Geometry*. UK, Cambridge University Press.
- Brown, D.P., Porta, H., Uhl, J. (1991). *Calculus and Mathematica*. Addison-Wesley Publishing Company.
- Devitt, J.S. (1993). *Calculus with Maple 5*. California, Brooks/Cole Publishing Company.
- Dreyfus, T. (1991). Advanced mathematical thinking processes. In Tall, D. (Ed.), *Advanced Mathematical Thinking*, London, Kluwer Academic Publishers, 25–41.
- Dreyfus, T., Hillel, J., Sierpinska, A. (1999). Cabri based linear algebra: transformations. In Schwank, I. (Ed.), *European Research in Mathematics Education I*, Osnabrueck, Forschungsinstitut fuer Mathematikdidaktik, 213–225.
- Dubinsky, E., Tall, D. (1991). Advanced mathematical thinking and the computer. In Tall, D. (Ed.), *Advanced Mathematical Thinking*, London, Kluwer Academic Publishers, 231–250.
- Gans, D. (1969). *Transformations and geometries.* New York, Appleton-Century-Crofts, Meredith Corporation..
- Hejny, M., Jirotkova, D., Stehlikova, N. (1997). *Geometricke transformace (metoda analyticka)*. Praha, PedF UK.
- Hejný, M., Kurina, F. (2001). *Díte, skola a matematika. Kosntruktivistické prístupy k vyucování*. Praha, Portál.
- Jaworski, B. (1994). *Investigating Mathematics Teaching: A Constructivist Enquiry*. London, The Falmer Press.
- Parks, J.M. (1997). Identifying Transformations by Their Orbits. In King, J., Schattschneider, D. (Eds.), *Geometry turned on. Dynamic software in learning, teaching, and research*. USA, The Mathematical Association of America, 105–108.
- Schattschneider, D. (1997). Visualization of Group Theory Concepts with Dynamic Geometry Software. In King, J., Schattschneider, D. (Eds.), *Geometry turned on. Dynamic software in learning, teaching, and research*. USA, The Mathematical Association of America, 121–127.
- *International Conference on the Teaching of Mathematics*. Proceedings. Samos, Greece, John Wiley & Sons, inc.

# Appendix

$$\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) & k\cos(\alpha)+u-u\cos(2\alpha)-v\sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) & k\sin(\alpha)+v+v\cos(2\alpha)-u\sin(2\alpha) \\ 0 & 0 & 1 \end{bmatrix}$$

**Figure 1**

$$G := \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) & m \\ \sin(2\alpha) & -\cos(2\alpha) & n \\ 0 & 0 & 1 \end{bmatrix}$$

**Figure 2**

- By comparing the two matrices we get the system of equations:

$$m = u\,(1-\cos(2\alpha)) - v\sin(2\alpha) + k\cos(\alpha),\ n = v\,(1+\cos(2\alpha)) - u\sin(2\alpha) + k\sin(\alpha)$$

- We will solve it in terms of $v$ and $k$.

```
> solve({m=u*(1-cos(2*alpha))-v*sin(2*alpha)+k*cos(alpha),n=v*(1+cos(2*alpha))-
  u*sin(2*alpha)+k*sin(alpha)},{v,k});
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$$\left\{ k = n\sin(\alpha) + \cos(\alpha)\,m,\ v = \frac{1}{2}\,\frac{n\sin(\alpha)^2 - n - 2\,u\sin(\alpha)\cos(\alpha) + \sin(\alpha)\cos(\alpha)\,m}{\sin(\alpha)^2 - 1} \right\}$$

```
>map(combine,(solve({m=u*(1-cos(2*alpha))-*sin(2*alpha)+k*cos(alpha),n=v*(1+cos(2*alpha))-
  u*sin(2*alpha)+k*sin(alpha)},{v,k})));
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$$\left\{ k = n\sin(\alpha) + \cos(\alpha)\,m,\ v = \frac{n + n\cos(2\alpha) + 2\,u\sin(2\alpha) - m\sin(2\alpha)}{2 + 2\cos(2\alpha)} \right\}$$

- We can see immediately that number $k$ equals the number $d$ which is a determining factor for a matrix to be a matrix of a line reflection or glide reflection. It remains to be seen how the equation of an axis can be found.
- Remember that $u$, $v$ are co-ordinates of any points on the line of reflection. Therefore if we write $x$ instead of $u$ and $y$ instead of $v$ in the above expression for $v$, we must get an equation of

$$\left\{ y = \frac{1}{2}\,\frac{n\cos(\alpha) + 2\,x\sin(\alpha) - \sin(\alpha)\,m}{\cos(\alpha)},\ k = n\sin(\alpha) + \cos(\alpha)\,m \right\}$$

the axis.

The equation of the line of reflection is: $x\sin\alpha - y\cos\alpha + \dfrac{n\cos\alpha - m\sin\alpha}{2} = 0.$

- The co-ordinates of the vector are: $\mathbf{u}[k\cos\alpha, k\sin\alpha]$, where $k = n\sin\alpha + m\cos\alpha$.

**Figure 3**

Constructivist teaching is based on constructivist learning theory. Constructivist teaching is based on the belief that learning occurs as learners are actively involved in a process of meaning and knowledge construction as opposed to passively receiving information. Constructivist approach teaching methods are based on constructivist learning theory. John Dewey and Jean Piaget researched childhood development and education; both were very influential in the development of informal education. Dewey's